Weighted Chairman Assignment and Flow-Time Scheduling

Victor Reis

joint work with
Siyue Liu

to appear in ITCS 2026

Microsoft Research

University of Washington Theory Seminar
November 18, 2025

Warmup I: discrepancy of two permutations

$$\begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 6 & 7 & 4 & 5 & 2 & 3 & 1 \end{array}$$

Goal: Color numbers red and blue so that $|\#\text{red} - \#\text{blue}| \leq 1$ in all $2n$ prefixes

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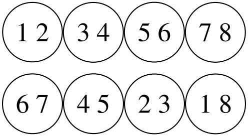

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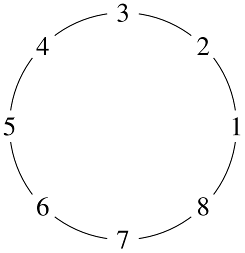

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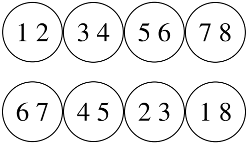

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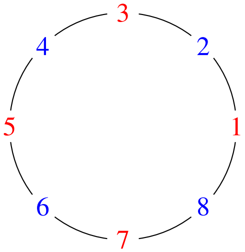

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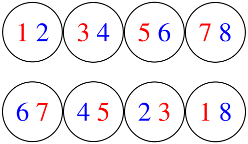

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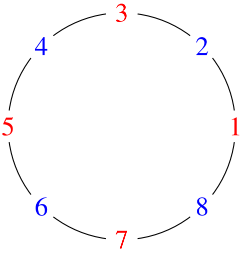

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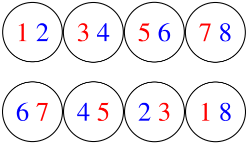

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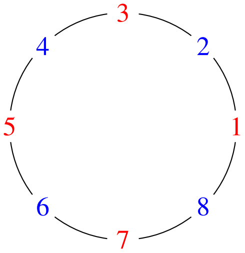

Open problem

Given *weights* $w \in [0,1]^n$ can we get $|w(\text{red}) - w(\text{blue})| \leq O(1)$ in all $2n$ prefixes?

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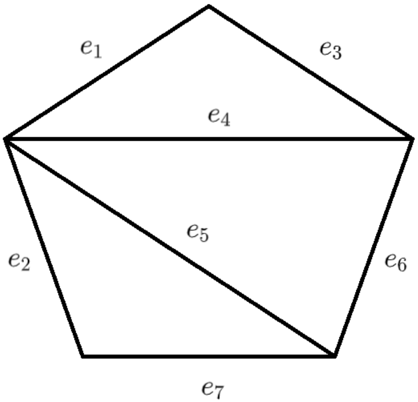

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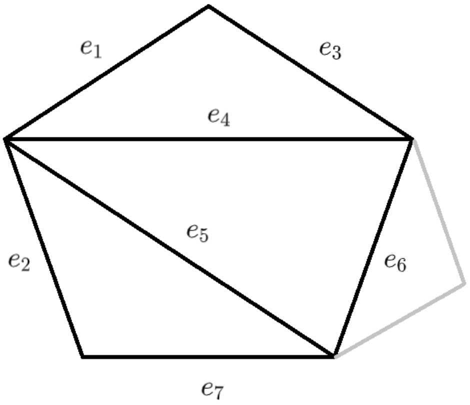

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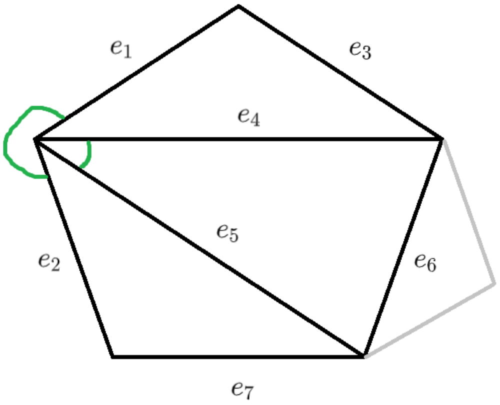

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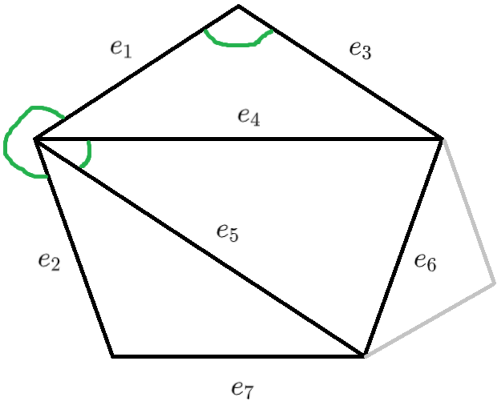

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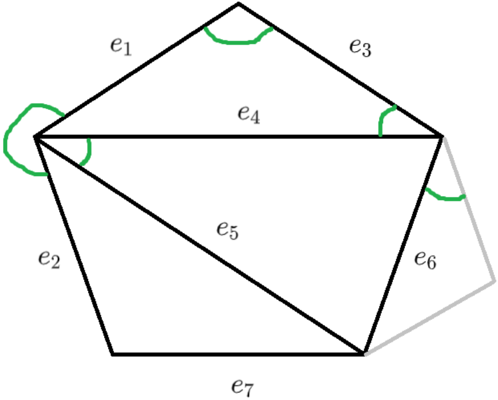

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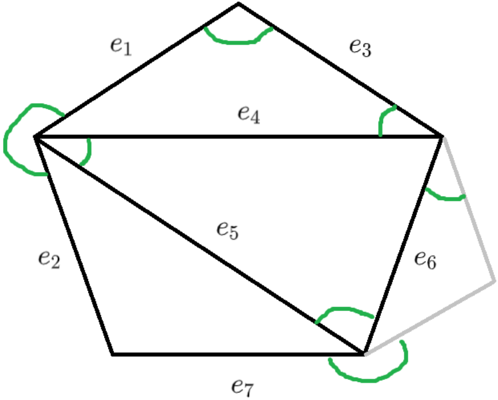

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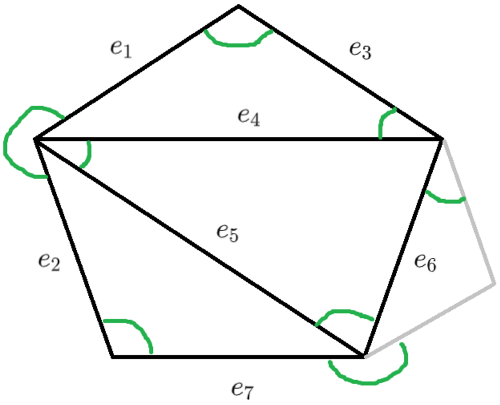

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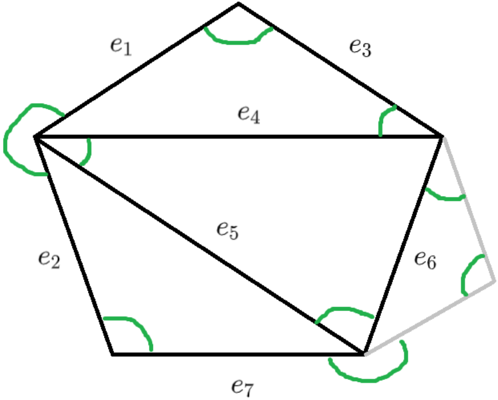

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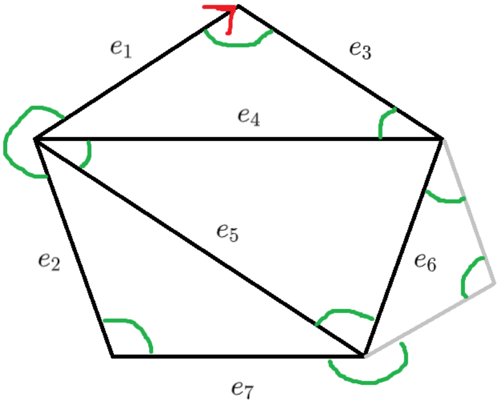

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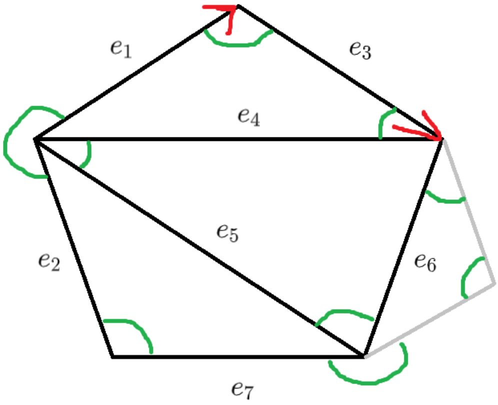

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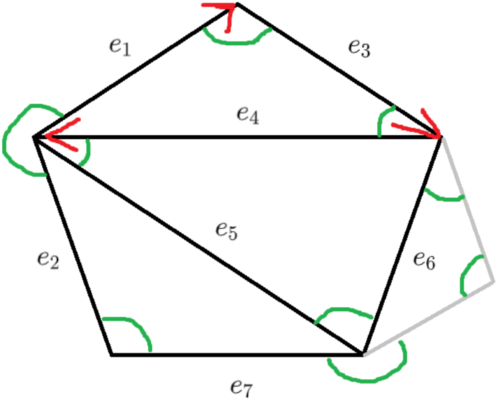

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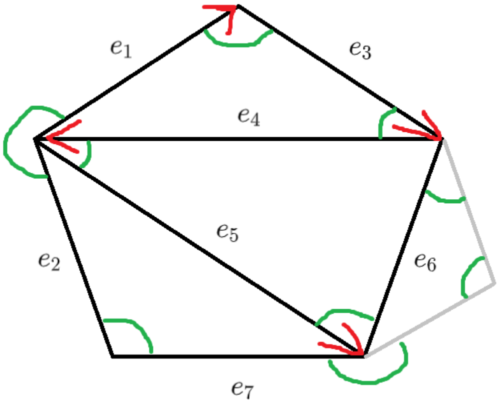

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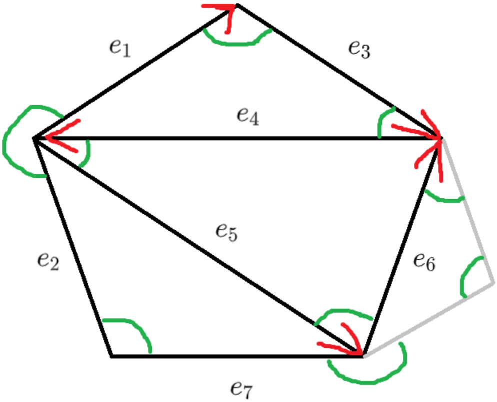

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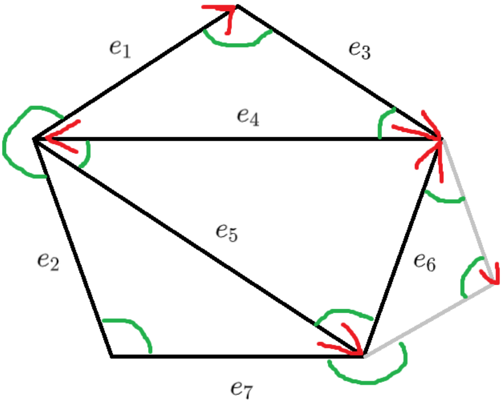

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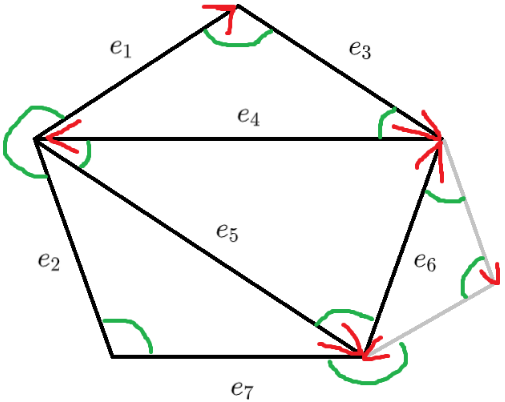

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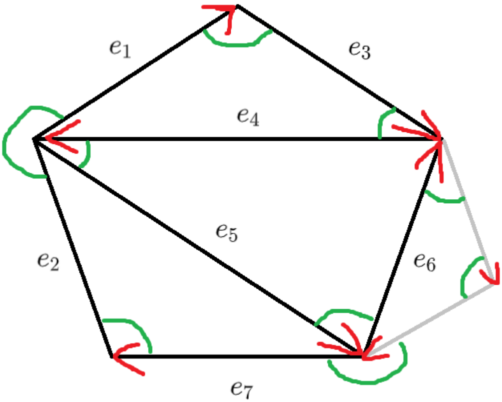

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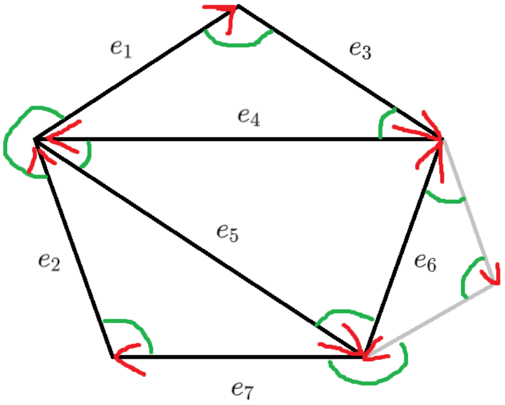

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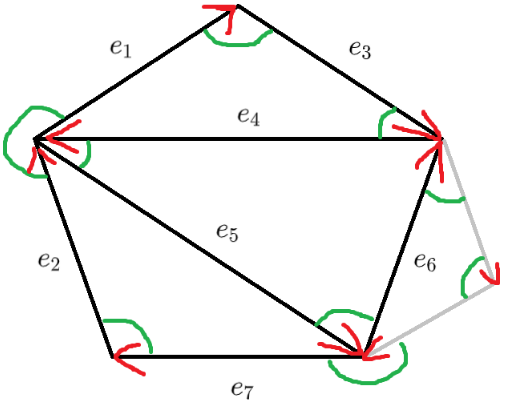

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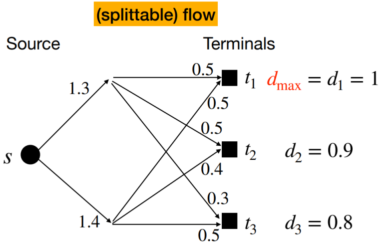
Single-source unsplittable flow (SSUF)



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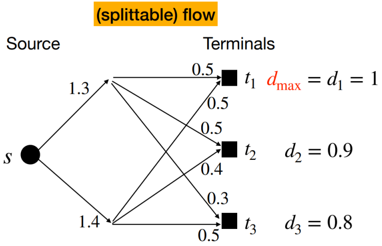

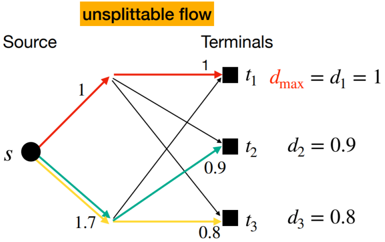

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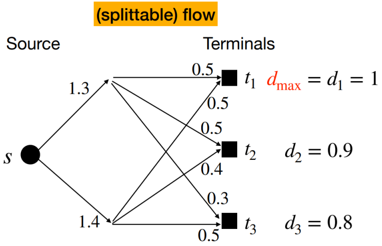

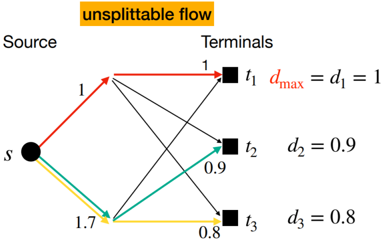

Theorem (Dinitz, Garg, Goemans, 1999)

Given directed $D = (V, E)$ and flow $x \in \mathbb{R}^E_{\geq 0}$, there exists an *unsplittable* flow $y \in \mathbb{R}^E_{\geq 0}$ such that $y(e) \leq x(e) + d_{\max}$ $\forall e \in E$.

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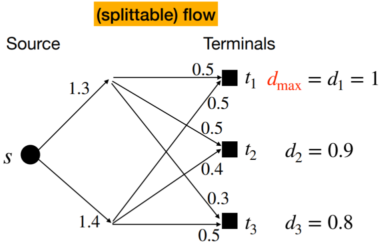

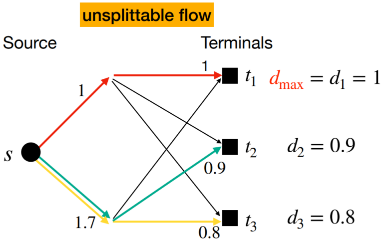

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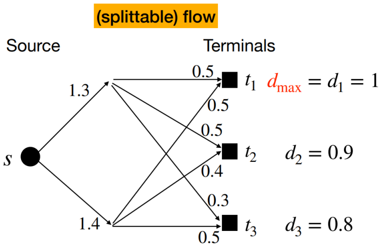

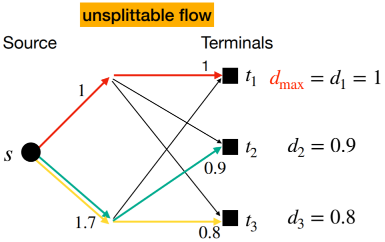

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- Known to imply a *cost-augmented* one-sided bound conjectured by Goemans [Swamy, Traub, Koch, Zenklusen, 2025]

Application in scheduling: makespan minimization

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\begin{aligned}
\min \quad & T \\
\text{s.t.} \quad & \sum_{i \in M} x_{ij} = 1, && \forall j \in J, \\
& \sum_{j \in J} d_j x_{ij} \leq T, && \forall i \in M, \\
& x \geq 0, \\
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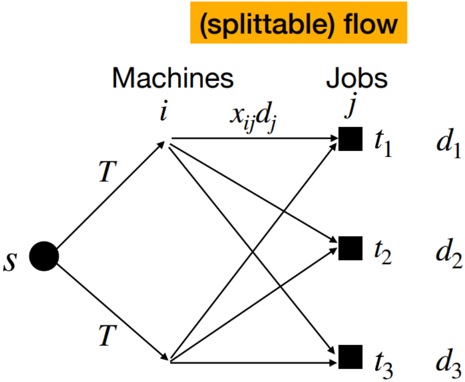

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unsplittable flow

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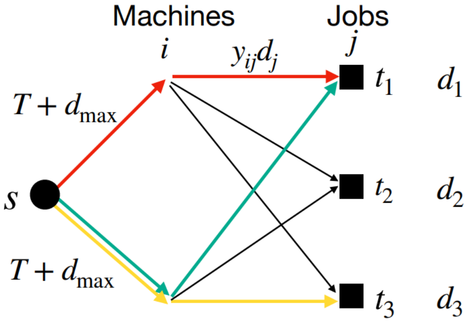

$$\sum_{j \in J} d_j y_{ij} \le \sum_{j \in J} d_j x_{ij} + d_{\max} \le OPT + OPT = 2OPT$$

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- Why is this an LP relaxation? Suppose we have an integral solution x

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- By time r_j, $\leq (r_j - r_a + T) - |[r_a, r_j]| = T$ volume left, so j finishes by $r_j + T$

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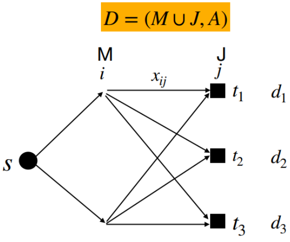

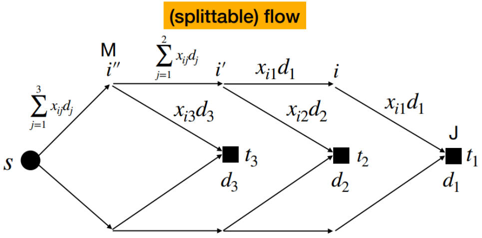

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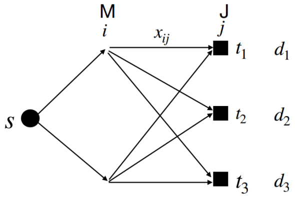
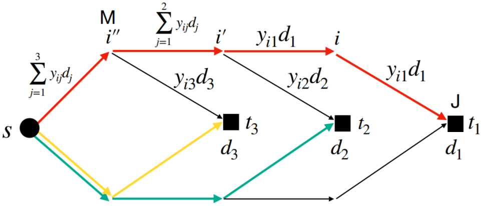


unsplittable flow



Application in scheduling II: flow-time minimization

Special case of Morell-Skutella's conjecture

Given bipartite $D = (M \cup J, E)$, and a fractional assignment $x \in [0,1]^{m \times n}$, there is an assignment $y \in \{0,1\}^{m \times n}$ such that

$$\left| \sum_{j=1}^{t} d_j x_{ij} - \sum_{j=1}^{t} d_j y_{ij} \right| \leq d_{\max}, \qquad \forall i \in [m],\ t \in [n].$$

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- Given a feasible LP solution x, can round it to y so that for any $1 \leq a \leq b \leq n$,

$$\begin{aligned}\sum_{j=a}^{b} d_j y_{ij} = \sum_{j=1}^{b} d_j y_{ij} - \sum_{j=1}^{a-1} d_j y_{ij} &\leq \left(\sum_{j=1}^{b} d_j x_{ij} + d_{\max} \right) - \left(\sum_{j=1}^{a-1} d_j x_{ij} - d_{\max} \right) \\ &= \sum_{j=a}^{b} d_j x_{ij} + 2d_{\max} \\ &\leq (r_b - r_a) + OPT + 2d_{\max} \\ &\leq (r_b - r_a) + 3OPT.\end{aligned}$$

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- Known up to $O(\sqrt{\log n})$ [Bansal, Svensson, Rohwedder STOC 2022]

Main results

Theorem (Liu, R., 2026)

Given a fractional assignment $x \in [0,1]^{m\times n}$ and weights $d \in \mathbb{R}^n_{>0}$, there exists an assignment $y \in \{0,1\}^{m\times n}$ so that for every $i \in [m]$ and $t \in [n]$,

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- The RHS can be improved to $(1 - \frac{1}{2m-2}) \cdot \max_{j\in[n]} d_j$, which is tight:

$$\begin{array}{c|cccc|}
 & 1 & 2 & \cdots & m-1 \\
\hline
1 & \frac{1}{2m-2} & \frac{1}{m-1} & \cdots & \frac{1}{m-1} \\
2 & \frac{1}{2m-2} & \frac{1}{m-1} & \cdots & \boxed{\frac{1}{m-1}} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
m-1 & \frac{1}{2m-2} & \boxed{\frac{1}{m-1}} & \cdots & \frac{1}{m-1} \\
m & \boxed{\frac{1}{2}} & 0 & \cdots & 0 \\
\hline
\end{array}$$

Application to flow-time scheduling

Corollary (Liu, R., 2026)

A 3-approx. algorithm for maximum flow-time minimization on identical machines.

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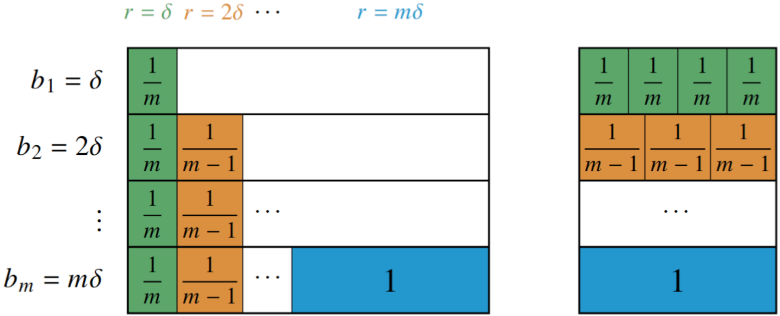

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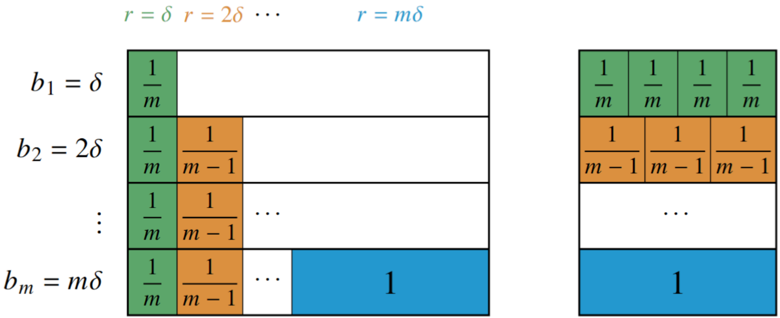

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$$P_t(i) := \sum_{j=1}^{t} d_j x_{ij}, \quad N_t(i) := \sum_{j=1}^{t} d_j y_{ij}, \quad \Delta_t(i) := P_t(i) - N_t(i).$$

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Algorithm 1 EARLIEST DEADLINE ALGORITHM

for $t \in [n]$ **do**
  $C(t) \leftarrow \{i \in [m] : P_t(i) > N_{t-1}(i)\}$ ▹ set of candidates at time t
  for $i \in [m]$ **do**
    $D_t(i) \leftarrow \min\{T \geq t : P_T(i) \geq N_{t-1}(i) + 1\}$ ▹ deadline of i at time t
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  Choose $s_t \in \arg\min_{i \in C(t)} D_t(i)$
  $y_{s_t,t} \leftarrow 1$ and $y_{i,t} \leftarrow 0$ for $i \in [m] \setminus \{s_t\}$
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Lower bound proof

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&\geq P_j(i) - N_{j-1}(i) - d_j \\
&> -d_j \geq -1.
\end{aligned}$$

Upper bound proof I

Given fractional assignment $x \in [0,1]^{m \times n}$, $d \in (0,1]^n$, define

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For any $i \in [m]$ and $t \in [n]$, $\Delta_t(i) < 1$.

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We will need to define another time $t^- \leq t' \leq t$ and analyze what happens in $(t', t]$.

Upper bound proof II

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$$P_t(s_{t^-}) < N_t(s_{t^-})$$

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t' := $\max\{j \le t : P_t(s_j) < N_{j-1}(s_j) + 1\}$ is well-defined.

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$$\begin{aligned} P_t(s_{t^-}) &< N_t(s_{t^-}) \\ &= N_{t^-}(s_{t^-}) \end{aligned}$$

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- $s_{t'} \notin I, s_j \in I \quad \forall j \in (t', t]$.
- $I \ne \emptyset$ since $I^+ \subseteq I$: $P_t(i) \ge N_t(i) + 1 \ge N_{t'-1}(i) + 1$ for any $i \in I^+$

Upper bound proof (last part!)

- Define $I := \{i \in [m] : P_t(i) \geq N_{t'-1}(i) + 1\} = \{i \in [m] : D_{t'}(i) \leq t\}$.
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For any $i \in I$ we have $P_t(i) \geq N_t(i)$ and $P_{t'}(i) \leq N_{t'-1}(i)$.

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Contradiction assuming Claim 3.2.:

$$\sum_{j \in (t', t]} d_j < 1 + \sum_{j \in (t', t]} d_j$$

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For the second inequality: $D_{t'}(s_{t'}) > t \implies I \cap C(t') = \emptyset$. □

Other conjectures

Non-uniform weights

Let $d \in \mathbb{R}_{>0}^{m\times n}$. For every fractional assignment $x \in [0,1]^{m\times n}$, there is an assignment $y \in \{0,1\}^{m\times n}$ so that for every $i \in [m]$ and $t \in [n]$,

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Weighted committee assignment

Let $d \in \mathbb{R}_{>0}^{n}$. For every $x \in [0,1]^{m \times n}$ so that $n_j := \sum_{i \in [m]} x_{ij} \in \mathbb{N}$ for all $j \in [n]$, there is $y \in \{0,1\}^{m \times n}$ so that $\sum_{i \in [m]} y_{ij} = n_j$ for all $j \in [n]$ and

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Thanks for your attention!